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Solvable spin-1 models in one dimension

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Abstract. We look for solvable spin-1 $SU(3)$-models in the seven-parameter manifold of translation- and reflection-invariant Hamiltonians with nearest-neighbour couplings. We prove that only discrete solutions of the Reshetikhin criterion exist. Therefore the situation here differs greatly from the corresponding spin-$\frac{1}{2}$ $SU(2)$ case, where Reshetikhin's criterion is satisfied for any model of the XYZ type.

1. Introduction

The spin-$\frac{1}{2}$ XYZ-model with Hamiltonian

$$H = \sum_{x=1}^{N} H(x, x+1) \tag{1.1}$$

and nearest-neighbour coupling

$$H(x, x+1) = \sum_{k=1}^{3} J_k \sigma_k(x) \sigma_k(x+1) \tag{1.2}$$

is known to be solvable for all values of the anisotropy parameters $J_k, k = 1, 2, 3$ [1] in the following sense. There is a transfer matrix with a spectral parameter from which one can derive an infinite set of commuting conservation laws. The existence of the first conserved local operator [2]

$$F_3 = \sum_{x=1}^{N} [H(x, x+1), H(x+1, x+2)] \tag{1.3}$$

can be proven directly (cf appendix A) without going through the machinery of inverse scattering transform methods [3, 4]. The key to this proof is hidden in the commutation relation:

$$\begin{aligned} [[H(x, x+1), H(x+1, x+2)], H(x, x+1) + H(x+1, x+2)] \\ = Q(x, x+1) - Q(x+1, x+2) \end{aligned} \tag{1.4}$$

for the building blocks $H(x, x+1)$ of the Hamiltonian. $Q(x, x+1)$ is some operator acting only on the spins at sites $x, x+1$. Equation (1.4) is Reshetikhin's criterion [4] which is

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necessary, but not sufficient for the existence of a solution of the Yang–Baxter equations. In any case (1.4) is sufficient for the validity of the first conservation law:

$$[H, F_3] = 0. \tag{1.5}$$

In this paper we are looking for solvable spin-1 models. The manifold of translation- and reflection-invariant Hamiltonians [5] that we are going to investigate is constructed from the nearest-neighbour couplings:

$$H(x, x+1) = \sum_{A=1}^{8} J_A \lambda_A(x) \lambda_A(x+1) \tag{1.6}$$

between the generators $\lambda_A(x)$, $A = 1, \ldots, 8$ of the group $SU(3)$. The generators—the Gell-Mann matrices—form a complete set of Hermitian and traceless 3×3 matrices which close under commutation:

$$[\lambda_A, \lambda_B] = 2\mathrm{i} \sum_{C=1}^{8} f_{ABC} \lambda_C \tag{1.7}$$

and anticommutation:

$$\{\lambda_A, \lambda_B\} = \tfrac{4}{3}\delta_{AB} + 2\sum_{C=1}^{8} d_{ABC}\lambda_C. \tag{1.8}$$

The structure constants d_{ABC} and f_{ABC} are totally symmetric and antisymmetric, respectively. They are listed in table 1. From a group-theoretical point of view the nearest-neighbour coupling (1.6) is the obvious spin-1 extension of the spin-$\frac{1}{2}$ XYZ-coupling (1.2). Three $O(3)$-invariant spin-1 models are known to be solvable in the sense explained above:

Table 1. The independent, non-vanishing components of f_{abc} and d_{abc}.

abc	f_{abc}	abc	d_{abc}	abc	d_{abc}
123	1	118	$1/\sqrt{3}$	366	$-1/2$
147	1/2	146	1/2	377	$-1/2$
156	$-1/2$	157	1/2	448	$-1/(2\sqrt{3})$
246	1/2	228	$1/\sqrt{3}$	558	$-1/(2\sqrt{3})$
257	1/2	247	$-1/2$	668	$-1/(2\sqrt{3})$
345	1/2	256	1/2	778	$-1/(2\sqrt{3})$
367	$-1/2$	338	$1/\sqrt{3}$	888	$-1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2		
678	$\sqrt{3}/2$	355	1/2		

(i) The $SU(3)$ symmetric Lai–Sutherland model [6] where the fundamental representations 3 of the generators reside at even and odd sites. This model is known to be gapless for an antiferromagnetic coupling. It is characterized by the couplings:

$$J_A = J \qquad A = 1, \ldots, 8. \tag{1.9}$$

(ii) The $SU(3)$ symmetric model where the fundamental representations 3 sit on the even sites and the complex conjugate representations 3^* sit on the odd sites [7]. It is characterized by the couplings:

$$J_A = J \qquad A = 1, 3, 4, 6, 8 \qquad \text{and} \qquad J_A = -J \qquad A = 2, 5, 7. \tag{1.10}$$

This model has been proven to have a gap [8] for $J > 0$.

(iii) The model of Kulish and Sklyanin [4, 9] is defined by the couplings

$$J_A = J \qquad A = 1, 3, 4, 6, 8 \qquad \text{and} \qquad J_A = -3J \qquad A = 2, 5, 7. \tag{1.11}$$

This model is known to be gapless for an antiferromagnetic coupling.

The three models are $O(3)$-invariant, since the Gell-Mann matrices $\lambda_2, -\lambda_5, \lambda_7$ form an $O(3)$ subalgebra. It has been proven recently by Kennedy [10] that these are the only $O(3)$-invariant spin-1 models which satisfy the Reshetikhin criterion (1.4). In this context it is remarkable to note that the solvability of the spin-$\frac{1}{2}$ models (1.1), (1.2) is not destroyed by breaking the $SU(2)$ symmetry. Therefore one would expect to find a one-(or more) parameter family of spin-1 Hamiltonians of the type (1.6) which are solvable in the sense explained above. Surprisingly enough, this is not the case. We will show in this paper that there are only eight non-trivial Hamiltonians of the type (1.6) which satisfy Reshetikhin's criterion (1.4). The corresponding couplings J_A are listed in table 3. The first three Hamiltonians are identical to the known $O(3)$-invariant ones.

The outline of the paper is as follows. In section 2 we demonstrate how the Reshetikhin criterion selects out the spin-1 Hamiltonians. In section 3 we derive a second criterion which has to be satisfied by a Hamiltonian in order to be completely integrable. In section 4 we compare our results with other solvable spin-1 models.

2. The Reshetikhin criterion for spin-1 models

In this section we are going to investigate how the Reshetikhin criterion (1.4) constrains the manifold of anisotropy parameters J_A in the nearest-neighbour couplings (1.6). For this purpose we have to evaluate the commutators on the left-hand side of (1.4) explicitly by means of (1.7), (1.8). This tedious computation leads to the following result:

$$\begin{aligned}[][H(1,2), H(2,3)], H(1,2) + H(2,3)] &= Q(1,2) - Q(2,3) \\ &\quad + R(1,2) + R(2,3) + S(1,3) + 4\sum_{ABF} T_{ABF}\lambda_A(1)\lambda_B(2)\lambda_F(3)\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}Q(1,2) &= \tfrac{8}{3}\sum_{ABC} J_A J_B^2 f_{ABC}^2 \lambda_A(1)\lambda_A(2) \\ &\quad + \tfrac{1}{\sqrt{3}}(J_4^2 + J_5^2 - J_6^2 - J_7^2)(J_3 + J_8)(\lambda_3(1)\lambda_8(2) + \lambda_3(2)\lambda_8(1))\end{aligned} \tag{2.2}$$

$$R(1,2) = \tfrac{1}{\sqrt{3}}(J_4^2 + J_5^2 - J_6^2 - J_7^2)(J_3 - J_8)(\lambda_3(1)\lambda_8(2) - \lambda_3(2)\lambda_8(1)) \tag{2.3}$$

$$S(1,3) = \tfrac{4}{\sqrt{3}}(J_4 J_5 - J_6 J_7)(J_8 - J_3)(\lambda_3(1)\lambda_8(3) - \lambda_8(1)\lambda_3(3)) \tag{2.4}$$

and

$$\begin{aligned}T_{ABF} &= J_A \sum_{CDE} J_D J_E f_{AEC}(d_{CDF} f_{EDB} + f_{CDF} d_{EDB}) \\ &\quad - J_B \sum_{CDE} J_D J_E f_{BEC}(d_{EDA} f_{CDF} + f_{EDA} d_{CDF}).\end{aligned} \tag{2.5}$$

The Reshetikhin criterion (1.4) is satisfied if (2.3), (2.4) and (2.5) vanish identically. From (2.3) and (2.4) we get either

$$J_3 = J_8 \qquad \text{or} \qquad (J_4 \pm J_5)^2 = (J_6 \pm J_7)^2. \tag{2.6}$$

From (2.5) we get

$$T_{ABF} = 0 \qquad A, B, F = 1, \ldots, 8 \tag{2.7}$$

Table 2. The non-trivial equations of (2.7).

1	$J_1[(J_4-J_5)^2-(J_6-J_7)^2]$	$=0$
2	$J_1[(J_4-J_5)^2+(J_6-J_7)^2+8(J_4J_5+J_6J_7)-8(J_2^2+J_3^2)]$ $+6J_8(J_4-J_5)(J_6-J_7)$	$=0$
3	$(J_4+J_5)[(J_4-J_5)(J_6-J_7)-4(J_1J_8-J_2J_3)]$	$=0$
4	$(J_1+J_2)[(J_1-J_2)(J_6-J_7)-(J_4-J_5)(5J_3-J_8)]$	$=0$
5	$(J_1+J_2)(J_4+J_5)(J_3-J_8)$	$=0$
6	$(J_4+J_5)[(J_1-J_2)(J_4-J_5)-(J_6-J_7)(5J_8-J_3)]$	$=0$
7	$(J_4+J_5)[(J_1-J_2)^2+(J_3-J_8)(J_3+3J_8)-(J_6-J_7)^2]$	$=0$
8	$(J_4+J_5)[(J_1+J_2)^2+(J_3+J_8)(J_3-3J_8)+4(J_6J_7-J_4J_5)]$	$=0$
9	$(J_4-J_5)[(J_1+J_2)^2+2J_8(J_3-3J_8)+4(J_6J_7+J_4J_5)]$ $+3(J_3+J_8)(J_1-J_2)(J_6-J_7)$	$=0$
10	$(J_4-J_5)[(J_1-J_2)^2+2(J_3-J_8)(J_3+3J_8)-(J_6-J_7)^2]$ $-3(J_3-J_8)(J_1-J_2)(J_6-J_7)$	$=0$
11	$J_3[(J_4-J_5)^2+4(J_6J_7+J_4J_5)-4(J_1^2+J_2^2)]$ $+J_8[3(J_4-J_5)^2+4(J_4J_5-J_6J_7)]$	$=0$

Equations (12) to (24) can be obtained by the transformations:
$J_4, J_5 \leftrightarrow J_6, J_7$ and/or $J_1, J_2 \leftrightarrow -J_2, -J_1$.

512 nonlinear equations for the eight couplings J_A. It turns out that most of these equations are trivially satisfied owing to the properties of the structure constants f_{ABC} and d_{ABC} listed in table 1. We are left with 24 non-trivial equations for the anisotropy parameters J_A. These remaining equations are listed in table 2. Most of the solutions of these equations lead to unitary equivalent Hamiltonians. At the end we found 10 solutions—listed in table 3—which are not obviously unitary equivalent. We checked for inequivalence by computing the traces:

$$\mathrm{tr}(H^2) = 4N\sum_A J_A^2 = 8N\,\mathrm{tr}_2 \tag{2.8}$$

$$\mathrm{tr}(H^3) = 4N\sum_{ABC}(d_{ABC}^2 - f_{ABC}^2)J_AJ_BJ_C = 16/3N\,\mathrm{tr}_3 \tag{2.9}$$

$$\begin{aligned}\mathrm{tr}(H^4) &= \frac{3(N-1)}{N}(\mathrm{tr}\,H^2))^2 + 64N\sum_{ACE} J_A^2J_C^2 d_{AAE}d_{CCE} \\ &\quad +4N\sum_{ABCDE} J_AJ_BJ_CJ_D(d_{ABE}^2 - f_{ABE}^2)(d_{CDE}^2 - f_{CDE}^2) \\ &= 192N(N-1)(tr_2)^2 + 32/9N\,\mathrm{tr}_4\,.\end{aligned} \tag{2.10}$$

These are also listed in table 3.

The Hamiltonians H_1 to H_5 are related to the 19-vertex model, which was recently investigated by Idzumi *et al* [12]. This connection will be discussed in section 4. Models H_6, H_7 and H_8 are candidates for new solvable spin-1 models unless they turn out to be unitarily equivalent to one of the Hamiltonians $H_1, \ldots, H_5$. Looking at the traces in table 3 one might suggest that H_6 is equivalent to H_5 and H_7, H_8 to H_2. However, we did not succeed in constructing the corresponding unitary operator. In appendix B we prove that H_9 is unitarily equivalent to H_1. The Hamiltonian H_{10} is trivially solvable as there is no interaction.

Finally let us point out that the coefficients in (2.5) are completely fixed by the symmetric and antisymmetric structure constants d_{ABC} and f_{ABC} of the group $SU(3)$. We also want to emphasize that Reshetikhin's criterion is trivially satisfied for all XYZ Hamiltonians (1.1), (1.2) with spin-$\frac{1}{2}$. For in the $SU(2)$ case, the symmetric structure constants d_{ABC}

Table 3. Solutions of the equations in table 2. A_1 and A_2 are constants.

Model	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	tr_2	tr_3	tr_4
H_1	1	1	1	1	1	1	1	1	4	−8	16
H_2	1	−3	1	1	−3	1	−3	1	16	136	1696
H_3	1	−1	1	1	−1	1	−1	1	4	28	196
H_4	0	0	0	1	1	1	1	0	2	0	32
H_5	1	−2	1	1	−1	1	−1	2	7	53	463
H_6	2	−1	1	1	−1	1	−1	2	7	53	463
H_7	3	−1	1	1	−3	1	−3	1	16	136	1696
H_8	1	−1	3	1	−3	3	−1	1	16	136	1696
H_9	−1	−1	1	1	1	1	1	1	4	−8	16
H_{10}	0	0	A_1	0	0	0	0	A_2			

are identically zero and therefore the $SU(2)$ analogue of (2.5), (2.7) is guaranteed for all anisotropy paramaters J_1, J_2, J_3. In this respect the spin-$\frac{1}{2}$ case is very special, which explains the large manifold of spin-$\frac{1}{2}$ models solvable with the Yang–Baxter equations.

3. A second criterion for the solvability of the Yang–Baxter equation

Reshetikhin's criterion results from an expansion of the Yang–Baxter equation in the spectral parameter of the R-matrix. This equation can be written in the form:

$$R_{12}(\lambda)R_{23}(\lambda+\mu)R_{12}(\mu) = R_{23}(\mu)R_{12}(\lambda+\mu)R_{23}(\lambda). \tag{3.1}$$

The non-singular R-matrix $R(\lambda)$ with spectral parameter λ is a one-parameter familiy of linear operators acting in the tensor product space $C^k \otimes C^k$. Here we are interested in the spin-1 case, i.e. $k = 3$. In equation (3.1) we have introduced the operators $R_{12}(\lambda)$ and $R_{23}(\lambda)$ which act in $C^k \otimes C^k \otimes C^k$. $R_{12}(\lambda)$ denotes $R(\lambda)$ acting on the first two factors and the identity operator on the third C^k:

$$R_{12}(\lambda) = R(\lambda) \otimes 1. \tag{3.2}$$

Correspondingly $R_{23}(\lambda)$ is defined as

$$R_{23}(\lambda) = 1 \otimes R(\lambda). \tag{3.3}$$

A solution of (3.1) is said to be regular if $R(0) = 1$, and a Hamiltonian with nearest-neighbour coupling

$$H = \sum_{x=1}^{N} H(x, x+1) \tag{3.4}$$

can be obtained via the expansion

$$R_{x,x+1}(\lambda) = 1 + \lambda H(x, x+1) + \sum_{n=2}^{\infty} \lambda^n R^{(n)}_{x,x+1}. \tag{3.5}$$

Any solution of the Yang–Baxter equation (3.1) therefore leads to an integrable Hamiltonian.

We now turn to the question of whether or not a given quantum spin Hamiltonian is integrable in the sense that it can be obtained from a regular solution of the Yang–Baxter equation. Substituting for each operator $R_{x,x+1}(\lambda)$ its power series in (3.1), we get several conditions which have to be satisfied by the coefficients $R^{(n)}_{x,x+1}$, $n = 1, 2, \ldots$.

The second-order term in the expansion (3.5) is given by

$$R^{(2)}_{x,x+1} = \tfrac{1}{2}H^2(x, x+1) + c_2 \tag{3.6}$$

with constant c_2. The third-order terms ($\lambda\mu^2$) turn out to be

$$\begin{aligned} R^{(3)}_{12} - R^{(3)}_{23} + \tfrac{1}{6}(H^3(2,3) - H^3(1,2)) + c_2(H(2,3) - H(1,2)) \\ = \tfrac{1}{6}[[H(1,2), H(2,3)], H(1,2) + H(2,3)]. \end{aligned} \tag{3.7}$$

This leads directly to Reshetikhin's criterion because there must exist an operator $Q(x, x+1)$ such that

$$[[H(1,2), H(2,3)], H(1,2) + H(2,3)] = Q(1,2) - Q(2,3). \tag{3.8}$$

This criterion is sufficient for the existence of the first (1.3) and second conserved operator:

$$\begin{aligned} F_4 = \sum_{x=1}^{N} Q(x, x+1) + \sum_{x=1}^{N}[[H(x, x+1), H(x+1, x+2)], H(x+1, x+2) \\ +2H(x+2, x+3)] \end{aligned} \tag{3.9}$$

$$[F_3, H] = [F_4, H] = 0. \tag{3.10}$$

At the fourth order in the expansion (3.5) we do not find any further restriction on the Hamiltonian. Terms proportional to $\lambda^2\mu^2$ and $\lambda\mu^3$ lead to

$$\begin{aligned} R^{(4)}_{x,x+1} = \tfrac{1}{24}H^4(x, x+1) + \tfrac{1}{2}c_2 H^2(x, x+1) + c_3 H(x, x+1) \\ +\tfrac{1}{12}\{H(x, x+1), Q(x, x+1)\} + c_4 \end{aligned} \tag{3.11}$$

with constants c_3, c_4.

A new criterion follows from the fifth order ($\lambda^2\mu^3$,$\lambda\mu^4$) in the expansion (3.5):

$$\begin{aligned} 10(R^{(5)}_{23} - R^{(5)}_{12}) - \tfrac{5}{3}c_2(H^3(2,3) + Q(2,3) - H^3(1,2) - Q(1,2)) \\ -\tfrac{1}{12}(H^5(2,3) - H^5(1,2)) - 10c_4(H(2,3) - H(1,2)) \\ -5c_3(H^2(2,3) - H^2(1,2)) - \tfrac{1}{6}(\{H^2(2,3), Q(2,3)\} - \{H^2(1,2), Q(1,2)\}) \\ +3H(2,3)Q(2,3)H(2,3) - 3H(1,2)Q(1,2)H(1,2) \\ = \tfrac{1}{12}Z(1,2,3). \end{aligned} \tag{3.12}$$

From (3.12) we conclude that there must exist an operator $\tilde{Q}(x, x+1)$ such that

$$Z(x, x+1, x+2) = \tilde{Q}(x, x+1) - \tilde{Q}(x+1, x+2). \tag{3.13}$$

The three-point function $Z(x, x+1, x+2)$ is defined as

$$\begin{aligned} Z(x, x+1, x+2) = [H(x, x+1) + H(x+1, x+2), X(x, x+1, x+2)] \\ +3[Q(x, x+1) + Q(x+1, x+2), [H(x, x+1), H(x+1, x+2)]] \end{aligned} \tag{3.14}$$

with

$$\begin{aligned} X(x, x+1, x+2) = [H(x, x+1), [H(x, x+1), [H(x, x+1), H(x+1, x+2)]]] \\ -[H(x+1, x+2), [H(x+1, x+2), [H(x+1, x+2), H(x, x+1)]]] \\ +\{\{H(x, x+1), H(x+1, x+2)\}, [H(x, x+1), H(x+1, x+2)]\}. \end{aligned} \tag{3.15}$$

Equation (3.12) implies that, in general, Reshetikhin's criterion is not sufficient for restoring the whole Yang–Baxter equation from the Hamiltonian. We have checked that the Hamiltonians H_6, H_7 and H_8 listed in table 3 satisfy the second criterion for integrability. We expect that at every odd order in the expansion of the Yang–Baxter equation a new criterion appears on the scene.

4. Further solvable spin-1 Hamiltonians

The manifold of spin-1 Hamiltonians is not exhausted by (1.1), (1.6). One can easily construct more general Hamiltonians. First of all one could think of adding terms which take into account the effect of uniform external fields:

$$H' = \sum_{x=1}^{N} (B_3\lambda_3(x) + B_8\lambda_8(x)) \tag{4.1}$$

which is the analogue to

$$H' = \sum_{x=1}^{N} B_3\sigma_3(x) \tag{4.2}$$

in the spin-$\frac{1}{2}$ case. The ansatz for the spin-1 case contains two terms corresponding to the two elements λ_3, λ_8 of the $SU(3)$-Cartan subalgebra. Indeed models with Hamiltonians $H + H'$—where H and H' are defined through (1.1), (1.6) and (4.1), respectively—have been investigated already. In terms of the nearest-neighbour couplings J_A in (1.6) and the field strength B_3, B_8 in (4.1), the solvable Fateev–Zamolodchikov [11] model has the following form:

$$\begin{aligned} &J_1 = J_3 = -\tfrac{1}{2} \qquad J_2 = \tfrac{3}{2} + 2\sinh^2\omega \qquad J_4 = J_6 = \tfrac{1}{2} - \cosh\omega \\ &J_5 = J_7 = \tfrac{1}{2} + \cosh\omega \qquad J_8 = -\tfrac{1}{2} - \tfrac{2}{3}\sinh^2\omega \\ &B_3 = 0 \qquad \text{and} \qquad B_8 = \tfrac{4}{3\sqrt{3}}\sinh^2\omega. \end{aligned} \tag{4.3}$$

Model (4.3) is usually written in terms of $SU(2)$ operators where the external field B_8 can be interpreted as a single-site anisotropy. Fateev and Zamolodchikov succeeded in constructing the transfer matrix for this one-parameter family of Hamiltonians. Note, however, that in these models the strength of the external field B_8 is related in a very special way to the strength of the nearest-neighbour couplings J_A. For $\omega = 0$ we find the $O(3)$-invariant model (1.11) of Kulish and Skylanin.

The model of Fateev and Zamolodchikov belongs to the set of 10 solutions of the Yang–Baxter equations for a 19-vertex model, as was recently shown in [12]. By imposing certain symmetries (among which is the ice-rule) on the R-matrices of the 19-vertex model the authors of [12] found a complete set of solutions of the Yang–Baxter equations. In terms of Gell-Mann matrices the corresponding spin-1 Hamiltonians are listed in table 4 ($B_3 = 0$ in all models). Model 1 is trivially solvable and includes H_{10} of table 3. The Hamiltonian H_4 in table 3 can be obtained from models 2 and 3 for special values of η. It was recently shown in [13] that they can be solved by mapping them to six-vertex models. The Lai–Sutherland model H_1 is included in model 4 and the Hamiltonians H_2, H_3, H_5 in table 3 correspond to models 7, 5 and 6, respectively.

All models in table 4 belong to the following class:

$$H = \sum_{x=1}^{N} \left(\sum_{A=1}^{8} J_A\lambda_A(x)\lambda_A(x+1) + B_8\lambda_8(x) \right) \tag{4.4}$$

where the couplings are restricted by $J_1 = J_3$, $J_4 = J_6$ and $J_5 = J_7$.

In terms of standard spin-1 matrices it is easy to show that they have the following properties in common:

(i) rotational invariance in the (x, y) plane,

(ii) invariance under $S^z \to -S^z$,

(iii) translation invariance and

Table 4. Solvable spin-1 Hamiltonians [12]: A_k are constants and $i_k = \pm 1$ for $k = 1, 2, 3, 4$. Model 10 is the Fateev–Zamolodchikov Hamiltonian (4.3).

Model	1	2	3	4	5	6	7	8	9
J_1	0	0	0	i_2	1	1	1	1	$1-\sqrt{5}$
J_2	A_1	0	0	i_1	-1	-2	-3	-1	$1+\sqrt{5}$
J_3	0	0	0	i_2	1	1	1	1	$1-\sqrt{5}$
J_4	0	1	1	i_4	1	1	1	$\frac{3}{\sqrt{2}}$	$2\left(\frac{-1+\sqrt{5}}{2}\right)^{1/2}$
J_5	0	1	1	i_4	-1	-1	-3	$-\frac{3}{\sqrt{2}}$	$-2\left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$
J_6	0	1	1	i_4	1	1	1	$\frac{3}{\sqrt{2}}$	$2\left(\frac{-1+\sqrt{5}}{2}\right)^{1/2}$
J_7	0	1	1	i_4	-1	-1	-3	$-\frac{3}{\sqrt{2}}$	$-2\left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$
J_8	A_2	$\frac{4}{3}\coth\eta$	0	$\frac{1}{3}(2i_3+i_1)$	1	2	1	$\frac{5}{3}$	$-\frac{5}{3}-\frac{1}{3}\sqrt{5}$
$12\sqrt{3}B_8$	A_3	$8\coth\eta$	$24\coth\eta$	$8(i_1-i_3)$	0	0	0	1	$4(5-2\sqrt{5})$

(iv) reflection invariance.

In order to find further exactly solvable models, one or more of these symmetries must be broken.

An example of a one-parameter family of solvable spin-1 models which violates reflection invariance has been given by Babelon *et al* [14] (see also [15]):

$$H_{\mathrm{BVV}} = H + J_{38}H_{38}. \tag{4.5}$$

The couplings J_A in H (cf (1.1), (1.6)) and J_{38} take the following values:

$$J_A = 1 \qquad A \neq 3, 8 \qquad J_A = \cosh\gamma \qquad A = 3, 8 \qquad J_{38} = -\frac{1}{\sqrt{3}}\sinh\gamma \tag{4.6}$$

$$H_{38} = \sum_{x=1}^{N}(\lambda_3(x)\lambda_8(x+1) - \lambda_8(x)\lambda_3(x+1)). \tag{4.7}$$

Grosse and Raschhofer [16] went one step further by giving up reflection and translation invariance. They found a family of solvable spin-1 Hamiltonians† with nearest-neighbour couplings

$$\begin{aligned} H(x, x+1) = {} & \cosh\gamma(\lambda_3(x)\lambda_3(x+1) + \lambda_8(x)\lambda_3(x+1)) \\ & + \frac{1}{\sqrt{3}}\sinh\gamma(\lambda_3(x)\lambda_8(x+1) - \lambda_8(x)\lambda_3(x+1)) \\ & + \cos\phi_1(x)(\lambda_1(x)\lambda_1(x+1) + \lambda_2(x)\lambda_2(x+1)) \\ & + \cos\phi_2(x)(\lambda_4(x)\lambda_4(x+1) + \lambda_5(x)\lambda_5(x+1)) \\ & + \cos(\phi_2(x) - \phi_1(x))(\lambda_6(x)\lambda_6(x+1) + \lambda_7(x)\lambda_7(x+1)) \\ & + \sin\phi_1(x)(\lambda_1(x)\lambda_2(x+1) - \lambda_2(x)\lambda_1(x+1)) \\ & + \sin\phi_2(x)(\lambda_4(x)\lambda_5(x+1) - \lambda_5(x)\lambda_4(x+1)) \\ & + \sin(\phi_2(x) - \phi_1(x))(\lambda_6(x)\lambda_7(x+1) - \lambda_7(x)\lambda_6(x+1)) \end{aligned} \tag{4.8}$$

and space-dependent couplings $\phi_1(x)$ and $\phi_2(x)$.

Finally, we would like to mention that there exist spin-1 Hamiltonians with exactly known ground-state properties [17], which were not derived as solutions of the Yang–Baxter

† We were informed by Raschhofer that the Hamiltonian quoted in [16] contains in addition an incorrect boundary term.

equation. Within the manifold of Hamiltonians (1.1), (4.4) these models are characterized by the following constraints on the couplings J_A and B_8:

$$\begin{aligned} J_1 &= J_3 \qquad J_4 = J_6 \qquad J_5 = J_7 \\ 2a^2 J_3 &= 3J_8 - 2J_1 - J_2 + 2\sigma(J_4 + J_5) \\ &= 2J_8 - \sqrt{3}B_8 + \sigma(J_4 + J_5) \\ &= a(J_5 - J_4) \end{aligned} \tag{4.9}$$

where a is a free parameter and $\sigma^2 = 1$. The ground states in these models can be represented as a matrix product of individual site states. The authors of [17] also give an additional criterion for the uniqueness of the ground state and the appearance of a gap (the 'Haldane scenario'). In terms of our couplings this criterion reads as follows:

$$a \neq 0 \qquad J_4 + J_5 \neq 0 \qquad J_1 + J_2 > 0 \qquad \text{and} \qquad J_1 > 0. \tag{4.10}$$

None of the models listed in table 4 satisfies both criteria (4.9) and (4.10).

5. Conclusions

In this paper we have investigated the seven-parameter manifold of spin-1 Hamiltonians with only nearest-neighbour couplings of the type (1.6). From a group theoretical point of view this manifold defines the natural extension of the spin-$\frac{1}{2}$ XYZ-model to the spin-1 case. Searching for solutions of the Yang–Baxter equation, we first looked for those spin-1 Hamiltonians satisfying the Reshetikhin criterion (1.4). The eight non-trivial Hamiltonians having this property are listed in table 3.

The first five are already known to be integrable. They are related to special cases of the 19-vertex model. The remaining three Hamiltonians are candidates for new solvable spin-1 models. They also passed a second criterion for the solvability of the Yang–Baxter equation.

Comparing the spin-$\frac{1}{2}$ and the spin-1 cases with nearest-neighbour couplings (1.2) and (1.6), respectively, we find a marked difference:

(i) The spin-$\frac{1}{2}$ XYZ Hamiltonians (1.2) are known to satisfy the Reshetikhin criterion for any choice of couplings J_k, $k = 1, 2, 3$.

(ii) On the seven-parameter manifold of spin-1 Hamiltonians (1.6) there is only a finite number of discrete solutions for the Reshetikhin criterion.

Continuous families of solvable spin-1 Hamiltonians have been constructed in [11, 12, 14] by adding further couplings. These families meet the seven-parameter manifold of spin-1 Hamiltonians (1.6) in those points where we find discrete solutions of the Reshetikhin criterion.

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Appendix A. The first conservation law and the Reshetikhin criterion

In this appendix we present a direct proof of (1.5) for the operator (1.3) provided that the Reshetikhin criterion (1.4) is satisfied. We split the Hamiltonian into four parts:

$$H = H_1 + H_2 + H_3 + H_4 \tag{A.1}$$

where

$$H_j = \sum_{x=j,j+4,\dots} H(x, x+1) \qquad j = 1, 2, 3, 4. \tag{A.2}$$

The four parts obey the following commutation relations:

$$[H_1, H_3] = [H_2, H_4] = 0 \tag{A.3}$$

$$[[H_j, H_{j+1}], H_j + H_{j+1}] = Q_j - Q_{j+1} \qquad j = 1, 2, 3, 4. \tag{A.4}$$

Equation (A.4) is a consequence of Reshekhitin's criterion (1.4) with

$$Q_j = \sum_{x=j,j+4,\dots} Q(x, x+1) \qquad j = 1, 2, 3, 4. \tag{A.5}$$

For any Hamiltonian which one can split into four parts satisfying the commutation relations (A.3), (A.4) the operator

$$F_3 = \sum_{j=1}^{4} [H_j, H_{j+1}] \tag{A.6}$$

is conserved:

$$[F_3, H] = \sum_{j=1}^{4} [[H_j, H_{j+1}], H_j + H_{j+1}] + \sum_{j=1}^{4} [[H_j, H_{j+1}], H_{j+2} + H_{j+3}]. \tag{A.7}$$

The commutators in the first term on the right-hand side add up to zero due to (A.4). The commutators in the second term add up to zero due to the Jacobi identity:

$$[[H_1, H_2], H_3]] + [[H_2, H_3], H_1] + [[H_3, H_1], H_2] = 0 \tag{A.8}$$

and due to (A.3).

Appendix B. Unitary equivalence of H_1 and H_9

In this appendix we want to prove that the Hamiltonian

$$H_9 = \sum_{x=1}^{N} \left(\sum_{A \neq 1,2} \lambda_A(x)\lambda_A(x+1) - \sum_{A=1,2} \lambda_A(x)\lambda_A(x+1) \right) \tag{B.1}$$

is unitarily equivalent to the Lai–Sutherland model

$$H_1 = \sum_{x=1}^{N} \left(\sum_{A=1}^{8} \lambda_A(x)\lambda_A(x+1) \right). \tag{B.2}$$

Let $|+\rangle$, $|-\rangle$ and $|0\rangle$ denote the three eigenstates of the operator λ_3:

$$\lambda_3|+\rangle = |+\rangle \qquad \lambda_3|-\rangle = -|-\rangle \qquad \text{and} \qquad \lambda_3|0\rangle = 0.$$

It is useful to define permutation operators P_{AB} acting on nearest-neighbour states of type $|A\rangle$ and $|B\rangle$ where $A, B = +, -, 0$. In terms of the Gell-Mann matrices, these permutations have the following form:

$$P_{+-} = \tfrac{1}{2} \sum_{x=1}^{N} (\lambda_1(x)\lambda_1(x+1) + \lambda_2(x)\lambda_2(x+1)) \tag{B.3}$$

$$P_{+0} = \tfrac{1}{2} \sum_{x=1}^{N} (\lambda_4(x)\lambda_4(x+1) + \lambda_5(x)\lambda_5(x+1)) \tag{B.4}$$

$$P_{-0} = \tfrac{1}{2} \sum_{x=1}^{N} (\lambda_6(x)\lambda_6(x+1) + \lambda_7(x)\lambda_7(x+1)) \tag{B.5}$$

$$P_{++} + P_{--} + P_{00} = \tfrac{1}{2} \sum_{x=1}^{N} (\lambda_3(x)\lambda_3(x+1) + \lambda_8(x)\lambda_8(x+1)) + \tfrac{1}{3}. \tag{B.6}$$

Here we have assumed that the three species $|+\rangle$, $|-\rangle$ and $|0\rangle$ are bosons. Fermions can be described by the transformation

$$P_{AA} \to -P_{AA}.$$

Following an argument from Sutherland [6] we now apply three unitary transformations U_1, U_2 and U_3 to the Hamiltonian

$$H = \tfrac{1}{2}H_1 + \tfrac{1}{3}N = \sum_{A,B=+,-,0} P_{AB}. \tag{B.7}$$

(i) Transformation U_1: For N even, proceed along the chain and multiply the state function at each even numbered site by -1 if the state is $|0\rangle$ and by $+1$ if the state is $|\pm\rangle$. For N odd we multiply the state function at each odd site by -1 and $+1$ if the state is $|0\rangle$ and $|\pm\rangle$, respectively. In both cases the effect of this transformation turns out to be

$$H \to U_1 H U_1^{-1} = H - 2(P_{+0} + P_{-0}). \tag{B.8}$$

(ii) Transformation U_2: Change the bosons into fermions by using a Jordan–Wigner transformation:

$$H \to U_2 U_1 H U_1^{-1} U_2^{-1} = -H + 2P_{+-}. \tag{B.9}$$

(iii) Transformation U_3: Multiply the state function by a function which is totally antisymmetric in all objects:

$$H \to U_3 U_2 U_1 H U_1^{-1} U_2^{-1} U_3^{-1} = H - 2P_{+-}. \tag{B.10}$$

From (B.3) we see that H_1 and H_9 are unitarily equivalent:

$$U_3 U_2 U_1 H_1 U_1^{-1} U_2^{-1} U_3^{-1} = H_9. \tag{B.11}$$

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